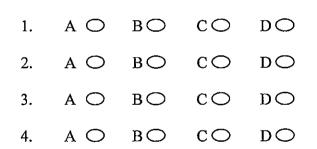# YEAR 12 MATHEMATICS ASSESSMENT TASK 3

June 2013

Topics Examined: Rates of Change, Exponentials, Logarithms, Exponential Growth and Decay

General Instructions

- Reading Time – 5 minutes

- Working Time – 60 Minutes

- Start each question on a new page.

Multiple Choice Q 1-4	
Q 5	
Q 6	
Q 7	
TOTAL	

Year 12 Mathematics Section I - Answer Sheet

Student Name/Number _____

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2   (B) 6   (C) 8   (D) 9

A ◯   B ⬤   C ◯   D ◯

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ⬤   B ✖   C ◯   D ◯

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ⬤   B ✖   C ◯   D ◯

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1.   A ◯   B ◯   C ◯   D ◯

2.   A ◯   B ◯   C ◯   D ◯

3.   A ◯   B ◯   C ◯   D ◯

4.   A ◯   B ◯   C ◯   D ◯

Section I   4 marks

Attempt Questions 1-4

Use the multiple choice answer sheet for Questions 1 – 4.

1. What is the value of $\int_2^6 \frac{1}{x+2}\ dx$

   (A) ln 2

   (B) ln 4

   (C) ln 6

   (D) ln 8

2. It is known that the number $N(t)$ of ants in a certain nest at time $t \geq 0$ is given by

   $N(t) = \frac{K}{1+e^t}$ where K is constant and t is measured in months

   At time $t = 0$, $N(t)$ is estimated at 2×10^5 ants.   What is the value of K ?

   (A) 2×10^5

   (B) 2×10^{-5}

   (C) 4×10^5

   (D) 4×10^5

3. e^{2lnx} is equal to

   (A) $\sqrt{x}$

   (B) x^2

   (C) $2x$

   (D) e^{2x}

4. The gradient of the tangent to the curve $y = e^{3x-6}$ at the point (2,1) is equal to

   (A) $\frac{1}{3}$

   (B) -3

   (C) $-\frac{1}{3}$

   (D) 3

QUESTION 5   12 Marks   Start your answer on a new page

(a)   Differentiate $\log_e\left(x^2 - 3x\right)$ (1)

(b)   Solve the equation $2\ln(x-5) = \ln(2x-7)$ (2)

(c)   Find the derivative of $\dfrac{e^{2x+1}}{2x+5}$ (2)

(d)   Evaluate $\log_2 9 = x$ giving your answer to 2 decimal places   (1)

(e)   Find $\dfrac{d}{dx}(2^x)$ (2)

(f)   Evaluate, leaving your answer in exact form $\displaystyle\int_0^2 e^{2x}\,dx$ (2)

(g) $\displaystyle\int \dfrac{x^2}{2x^3 - 7}\,dx$ (2)

QUESTION 6   14 Marks   Start your answer on a new page

(a)   Consider the function $f(x) = x - 2\log_e x$ (for $x>0$)

       i)   Find the first and second derivative of $f(x)$ (2)

       ii)   Find the co-ordinates of the turning point and determine its nature.   (2)

       iii)   Show that there are no points of inflection   (1)

       iv)   What is the maximum value of $x - 2\log_e x$ in the domain $1 \le x \le 5$ (2)

       v)   Hence sketch the curve $y = x - 2\log_e x$ for $1 \le x \le 5$ (1)

(b)   The surface area of a balloon is given by the formula $A = 2t^2 + 5t + 7$ where t is the time in seconds and A is the surface area in cm^2. Find the rate that the surface area is increasing after 5 seconds.   (2)
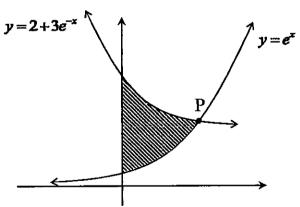

(c)   i)   Show that $\dfrac{d}{dx}(x\ln x - x) = \ln x$ (2)

       ii)   Hence, or otherwise, find $\displaystyle\int \ln x^2\,dx$ (2)

QUESTION 7   14 Marks   Start your answer on a new page

(a)   Following an accident, water started leaking out of a tank. If the volume of water in the
       tank was V litres, then t days after the accident, the rate of change of V was given by

       $\dfrac{dV}{dt} = 20t - 300$ litres per day. When the tank stopped leaking, it still had 4750 litres in it.

       i)   Show that the tank was leaking for 15 days   (1)

       ii)   Find a formula for V in terms of t.   (2)

       iii)   How much water was in the tank just before it started leaking?   (1)

(b)   A plane which is initially flying at a speed of 200 m/s cuts its engines. It then decelerates at
       a rate which is proportional to its speed at that time   (i.e. $\dfrac{dS}{dt} = -kS$).

       i)   Show that $S = S_0 e^{-kt}$ is a solution to this equation.   (1)

       ii)   After 50 seconds its speed is 120 m/s. Find the value of k (to 4 decimal places)   (2)

       iii)   Find the speed after 90 seconds correct to the nearest m/s   (2)

(c)

       The diagram shows the graphs of $y = e^x$ and $y = 2 + 3e^{-x}$ intersecting at the point P.

       i)   Show that the curves intersect when $e^{2x} - 2e^x - 3 = 0$ (1)

       ii)   Hence show that the x-coordinate of the point P is $\ln 3$ (2)

       iii)   Find the shaded area bound by the curves $y = e^x$, $y = 2 + 3e^{-x}$ and the y axis   (2)

STANDARD INTEGRALS

$$\int x^n\,dx \qquad = \frac{1}{n+1}x^{n+1}, \ n \neq -1; \ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx \qquad = \ln x, \ x > 0$$

$$\int e^{ax}\,dx \qquad = \frac{1}{a}e^{ax}, \ a \neq 0$$

$$\int \cos ax\,dx \qquad = \frac{1}{a}\sin ax, \ a \neq 0$$

$$\int \sin ax\,dx \qquad = -\frac{1}{a}\cos ax, \ a \neq 0$$

$$\int \sec^2 ax\,dx \qquad = \frac{1}{a}\tan ax, \ a \neq 0$$

$$\int \sec ax \tan ax\,dx \qquad = \frac{1}{a}\sec ax, \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx \qquad = \frac{1}{a}\tan^{-1}\frac{x}{a}, \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx \qquad = \sin^{-1}\frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx \qquad = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx \qquad = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \ x > 0$

1. $\log_e 8 - \log_e 4 = \log_e 2$

2. $2.2 \times 10^5 = \frac{k}{2}$

$4 \times 10^5 = k$ **(A)**

3. $e^{2\ln x} = e^{\ln x^2}$

$= x^2$ **(C)**

4. $y' = 3e^{3x-6}$

at $x=2$ $y' = 3e^0$

$= 3$ **(B)**

5. $\frac{2x-3}{x^2-3x}$ **(D)**

b) $2\ln(x-5) = \ln(2x-7)$

$(x-5)^2 = 2x-7$ ✓

$x^2 - 10x + 25 = 2x - 7$

$x^2 - 12x + 32 = 0$

$(x-8)(x-4) = 0$

$x = 8$ or $x = 4$

$x = 8$ only   ✓✓

c) $u = e^{2x+1}$ $v = 2x+5$

$u' = 2e^{2x+1}$ $v' = 2$

$\dfrac{(2x+5).2e^{2x+1} - 2e^{2x+1}}{(2x+5)^2}$ ✓✓

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d) $\dfrac{\log_e 9}{\log_e 2} = x$

$x = 3.17$ ✓   **(A)**

e) $\ln 2 . 2^x$ ✓✓   **(C)**

f) $\left[\frac{1}{2}e^{2x}\right]_0^2 = \frac{e^4-1}{2}$ ✓✓   **(B)**

g) $\frac{1}{6}\ln(2x^3-7) + c$ **(D)**

QUESTION 6

a) i) $f'(x) = 1 - \dfrac{2}{x}$ or $1-2x^{-1}$ ✓

$f''(x) = 2x^{-2}$ or $\dfrac{2}{x^2}$ ✓

ii) for turning point $f'(x)=0$

$1 - \dfrac{2}{x} = 0$

$x = 2$

$y = 2 - 2\ln2$ $(\doteq 0.6)$ **(1)**

or $f''(2) = \dfrac{2}{4}$ **(1)**

$= \dfrac{1}{2} > 0$ ☺

$\therefore$ min T.P

x	1	2	3
$f'(x)$	-1	0	$\frac{1}{3}$

$\diagdown\,\diagup$

minimum T.P

iii) $f''(x)=0 \Rightarrow \dfrac{2}{x^2}=0$

no solution

$\therefore$ no T.P

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at $x=1$ $y = 1-2\ln 1 = 1$

at $x=5$ $y = 5-2\ln5$

$\doteq 1.78$

$\therefore$ max value $5-2\ln5$ **(2)**

v)

b) $A = 2t^2 + 5t + 7$

$\dfrac{dA}{dt} = 4t + 5$ ✓

at $t = 5$

$\dfrac{dA}{dt} = 25$ ✓

c) $\dfrac{d}{dx}(x\ln x - x)$

$= 1.\ln x + x.\dfrac{1}{x} - 1$

$= \ln x + 1 - 1$

$= \ln x$

c) ii) $\ln x^2 = 2\ln x$

$\therefore 2\int \ln x \, dx = 2(x\ln x - x) + c$

QUESTION 7.

(a) $\frac{dV}{dt} = 0$ stops leaking

$0 = 20t - 300$

$t = 15$ days

ii) $V = \int 20t - 300$

$V = 10t^2 - 300t + c$

when $t = 15$ $V = 4750$

$4750 = 2250 - 4500 + c$

$c = 7000$

$V = 10t^2 - 300t + 7000$

iii) $V = 7000$.

b)i) $S = S_0 e^{-kt}$

$\frac{ds}{dt} = -kS_0 e^{-kt}$

$= -kS$

ii) when $t = 0$ $S_0 = 200$

$\therefore S = 200e^{-kt}$

ii) $120 = 200e^{-k \cdot 50}$

$\ln\left(\frac{120}{200}\right) = -50k$

$\frac{\ln(0.6)}{50} = -k$

$k = 0.0102$

iii) $S = 200e^{-0.0102 \times 90}$

$= 80$ m/s.

c) Solve simultaneously

$e^x = 2 + 3e^{-x}$

$e^x - 2 - 3e^{-x} = 0$ $\left(\text{Note } e^{-x} = \frac{1}{x}\right)$

$e^x - 2 - \frac{3}{e^x} = 0$

$e^{2x} - 2e^x - 3 = 0$

ii) let $u = e^x$

$u^2 - 2u - 3 = 0$

$(u-3)(u+1) = 0$

$u = 3$ or $u = -1$

$e^x = 3$ $e^x \neq -1$

$\therefore x = \ln 3$

iii) $\int_0^{\ln 3} 2 + 3e^{-x} - e^x \, dx$

$\left[2x - 3e^{-x} - e^x \right]_0^{\ln 3}$

$= \left(2\ln 3 - 3e^{-\ln 3} - e^{\ln 3} \right)$

$- (0 - 3.1 - 1)$

$= \left(2\ln 3 - 3e^{\ln \frac{1}{3}} - e^{\ln 3} \right)$

$- (-4)$

$= 2\ln 3 - 1 - 3 + 4$

$= 2\ln 3$